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THE PROBLEM OF TORSION OF A HOLLOW CIRCULAR CYLINDER WITH VARIABLE SHEAR MODULI†

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Classes of functions for shear moduli which depend on a cylindrical system of coordinates are singled-out and general solutions for the stress and displacement functions are constructed in the form of a finite integral operator consisting of completely defined variable coefficients and an arbitrary analytic function of a complex argument. An example of the solution of the problem with mixed boundary conditions of the torsion of a hollow cylinder, one end of which is rigidly clamped, is given.

1. FUNDAMENTAL EQUATIONS. GENERAL SOLUTION

THE TORSION of an anisotropic non-uniform solid of revolution in a cylindrical system of coordinates r, θ, z (where z is directed along the axis of symmetry) is characterized by the following linear system of elliptic-type equations [1, 2]

$$\frac{\partial\psi}{\partial r}-r^3G_1\frac{\partial\varphi}{\partial z}=0,\qquad \frac{\partial\psi}{\partial z}+r^3G_2\frac{\partial\varphi}{\partial r}=0 \tag{1.1}$$

Here ψ is the stress function, φ is the displacement function, and $G_{\theta z}=G_1(r)$, $G_{r\theta}=G_2(r)$ are the shear moduli.

Here, the force stresses $\tau_{\theta z}=\tau_1(r,z)$, $\tau_{r\theta}=\tau_2(r,z)$ and the displacement $u_r=v(r,z)$ are defined in terms of the functions ψ and φ by the formulae

$$\tau_1=\frac{1}{r^2}\frac{\partial\psi}{\partial r}=rG_1\frac{\partial\varphi}{\partial z},\qquad \tau_2=-\frac{1}{r^2}\frac{\partial\psi}{\partial z}=rG_2\frac{\partial\varphi}{\partial r},\qquad v=r\varphi \tag{1.2}$$

while the torque at the end $z=0$ is given by the formula

$$M=2\pi\int_{r_1}^{r_2}r^2\tau_1\,dr=2\pi[\psi(r_2,0)-\psi(r_1,0)] \tag{1.3}$$

For arbitrary shear moduli, which depend on the radius, a general solution of system (1.1) was constructed in [3] in the form of complex infinite integral and differential series.

We will single-out classes of functions for the shear moduli which enable us, for system (1.1), to construct a general solution in the form of a finite operator.

Henceforth we will consider the case when $r\neq0$ (a hollow cylinder).

2. CONSTRUCTION OF THE GENERAL SOLUTION

We will seek a solution of system (1.2) in the form of an integral operator

$$\begin{aligned}\psi&=\operatorname{Re}[\alpha(r)f(\zeta)+A\int f(\zeta)\,d\zeta]\\ \varphi&=\operatorname{Im}[\beta(r)f(\zeta)+B\int f(\zeta)\,d\zeta]\end{aligned} \tag{2.1}$$

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Here A and B are arbitrary constants, $\alpha(r)$, $\beta(r)$ are real functions of one argument r, and $f(\zeta)$ is an arbitrary analytic function of the complex variable

$$\zeta = \int \rho dr + iz \quad (\rho = \sqrt{G_1/G_2}) \tag{2.2}$$

We will construct the corresponding derivatives of expressions (2.1) and introduce them into the system of equations (1.1). As a result, using the well-known properties of analytic functions of a complex variable and dropping the signs Re and Im, we obtain the following two equations

$$\begin{aligned} &(\alpha' - Br^3 G_1 + A\rho) f + (\alpha\rho - r^3 G_1 \beta) f' = 0 \\ &[r^3 G_2 (\beta' + B\rho) - A] f + (r^3 G_2 \beta\rho - \alpha) f' = 0 \end{aligned} \tag{2.3}$$

which are satisfied, for an arbitrary function $f(\zeta)$, if we impose the following conditions on α, β

$$\alpha = B\int r^3 G_1 dr - A\int \rho dr, \quad \beta = A\int \frac{dr}{r^3 G_2} - B\int \rho dr \tag{2.4}$$

$$\alpha = r^3 \sqrt{G_1 G_2}\, \beta \tag{2.5}$$

In (2.4) we have omitted the additive constants of integration as being unimportant.

It can be shown by substitution that when $A = 1$ and $B = 0$, Eq. (2.5) is satisfied, for example, for the following shear moduli

$$1)\ G_1 = ar^p, \quad G_2 = br^q \quad (q = (p-2)/3) \tag{2.6}$$

$$2)\ G_1 = ar^{-3} e^{pr}, \quad G_2 = br^{-3} e^{qr} \quad (q = p/3) \tag{2.7}$$

$$3)\ G_1 = ar^{-4} \ln^2 r, \quad G_2 = br^{-2} \ln^2 r \tag{2.8}$$

Here a and b are constants, p is an arbitrary number (integer, fractional, positive or negative), which can be used when approximating the shear moduli.

When $A = 0$ and $B = 1$, condition (2.5) is satisfied for the moduli (2.6) when $q = 2p + 10$, while for the moduli (2.7) it is satisfied when $q = 3p$. For the functions (2.8), Eq. (2.5) cannot be satisfied.

When $A \neq 0$, $B \neq 0$, condition (2.5) is only satisfied for the function (2.6), where $p = -4$ and $q = -2$.

Equations (2.6)–(2.8) are suitable when $r \neq 0$, that is, for determining the stress state of hollow cylinders. For each case (2.6)–(2.8) the functions α, β ζ and φ, ψ (2.1) take a specific form.

Shear moduli of the form (2.6) when $p = q$ ($p = \pm 1$, $p = \pm 2$) and when $q = p + 2$ ($p \neq -4$ and $p = -4$) were considered in [2] when determining the stress functions in a different way in the form of series containing Bessel functions of the first and second kind of imaginary argument ir; cases were singled-out when they degenerate into hyperbolic or power functions with complex exponents.

If in (2.1) the function $f(\zeta)$ is specified arbitrarily in the form of a power or some other elementary analytic function of a complex variable, then, like their sums, they will all give a whole set of different inverse boundary value problems, some of which may turn out to be useful in practice.

To solve the direct boundary value problem, we will take the function $f(\zeta)$ in the form of a converging exponential series [4]

$$f(\zeta) = \Sigma n^{-1} (a_n e^{n\omega\zeta} + b_n e^{-n\omega\zeta}) \tag{2.9}$$

in which a_n and b_n are arbitrary real numbers (in general, complex), and ω is the characteristic number of the specific problem, determined from the boundary conditions. Here and henceforth the summation is taken from $n = -1$ to $n = \infty$.

3. A HOLLOW CYLINDER WITH MIXED BOUNDARY CONDITIONS ON THE SIDE OF SURFACES

Consider a non-homogeneous hollow circular cylinder of length l with internal radius r_1 and external radius r_2. Suppose that the displacement is specified on the internal surface, while the surface force is specified on the external surface. We will assume the end $z = l$ is rigidly clamped (a cantilever), while forces are applied to the free end ($z = 0$) leading to a torque (1.3).

We will introduce the function (2.9) into (2.1) ($A = 1$ and $B = 0$) and we will separate the real and imaginary parts. We obtain the following expressions

$$\psi = \Sigma n^{-1} [(\alpha + (n\omega)^{-1}) a_n e^{n\omega\rho} + (\alpha - (n\omega)^{-1}) b_n e^{-n\omega\rho}] \cos n\omega z \tag{3.1}$$
$$\varphi = \beta \Sigma n^{-1} (a_n e^{n\omega\rho} - b_n e^{-n\omega\rho}) \sin(n\omega z)$$

and the boundary conditions

$$v|_{r=r_1} = r_1 f_1(z), \quad \frac{\partial\psi}{\partial z}\Big|_{r=r_2} = f_2(z), \quad \frac{\partial\psi}{\partial r}\Big|_{z=0} = \frac{\partial\psi}{\partial r}\Big|_{z=l} = 0 \tag{3.2}$$

Note that if in (3.1) we change from exponential to hyperbolic functions, we obtain a result similar to that obtained earlier [2].

The last two conditions of (3.2) will be satisfied if we put $\omega = \pi/(2l)$ and $n = 1, 3, 5, \ldots$, or, which is the same thing, replace n by $2k+1$ ($k = 1, 2, \ldots$). The first two conditions in (3.2) will also be satisfied if the functions $f_1(z)$ and $f_2(z)$ are expanded in terms of $\sin\omega z$ in the range $0 \leqslant z \leqslant l$ and we then use the usual Fourier method. Doing this we obtain

$$a_n = D_+/\Delta, \quad b_n = D_-/\Delta, \quad \Delta = \Delta_+ + \Delta_- \tag{3.3}$$
$$D_\pm = -e^{\mp n\omega\rho_1} \left(\frac{B_n}{\omega} \mp \frac{nC_n}{\beta_1} \Delta_\mp\right)$$
$$\Delta_\pm = \left(\alpha_2 \pm \frac{1}{n\omega}\right) e^{\mp n\omega(\rho_1 - \rho_2)}$$
$$C_n = \frac{2}{l} \int_0^l f_1(z) \sin n\omega z dz, \quad B_n = \frac{2}{l} \int_0^l f_2(z) \sin n\omega z dz$$
$$\alpha_i = \alpha(r_i), \quad \rho_i = \rho(r_i) \quad (i = 1, 2), \quad \beta_1 = \beta(r_1)$$

Equations (3.1) and (3.3) naturally relate to an equal extent to any of the three combinations of shear moduli (2.6)–(2.8). No difficulties arise in calculating the torque (1.3) at the free end $z = 0$.

Note that if we introduce the notation $F(\zeta) = \int f(\zeta) d\zeta$ into (2.1) these functions take the form of a differential operator.

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